Chaos in sequence spaces

A Senior Comprehensive Project
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I hereby recognize and pledge to fulfill my responsibilities, as defined in the Honor Code, and to maintain the integrity of both myself and the College community as a whole.

Pledge:

Kodi Arfer

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Abstract

I discuss a 2003 paper by Alfredo Peris, making it accessible to undergraduates with some knowledge of analysis and topology by providing background material and more detailed proofs. The main theorems identify chaotic families of functions on the sequence spaces c_0 and ℓ_q. These functions can be expressed as a backward shift composed with an operator that applies a fixed complex-valued polynomial to each sequence element. The background material proves facts about c_0 and ℓ_q and complex dynamics.

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❶ Introduction

In the theory of dynamical systems, which is concerned with the behavior of a function as it's repeatedly applied to a topological space, we're often interested in identifying chaotic functions, that is, functions for which, in any of several precise senses, approximation is useless. Chaos is both a blessing and a curse. It's much of what makes dynamics mathematically interesting, but it also makes predicting the weather a month in advance impossible.

Here our concern is not a system with obvious real-world implications but spaces of sequences of complex numbers. These sequence spaces are among the simplest and analytically friendliest examples of infinite-dimensional normed vector spaces. As the standard examples of metric spaces more general than the real numbers, they serve as an ideal setting in which to generalize real analysis. Thus they're also a logical place to begin if we want to generalize complex dynamics. Examining operators on sequence spaces, we'll see that some of the dynamic properties of the underlying complex-valued polynomial can be lifted up to the sequence space. At the same time, the sequence-space operators can be *easier* to reason about than their underlying polynomials. For example, I couldn't tell you much about the dynamics of $z \mapsto 5z^7 + z^4 + 2z$, but Theorem 7.1 tells me that the sequence-space equivalent is chaotic.

Part of our success will come from incorporating into our sequence-space operators the backward shift operator, which takes a sequence and removes its first element, so that the nth element of the image is the $(n+1)$st element of the preimage. Surprisingly, the weighted backward shift operator, which multiplies each element by some constant $\lambda \in \mathbb{C}$, is chaotic for $|\lambda| > 1$ (Rolewicz, 1969). The value of the backward shift for us is that it allows us to construct sequences which have a sort of dual nature. We can choose the first few elements of a sequence in order to satisfy one set of conditions, and the rest of the sequence to satisfy some other conditions, and thanks to the shift, the prefix will eventually disappear. In particular, any sequence which is eventually constantly 0 eventually becomes the zero sequence when shifted enough times.

❷ Sequence spaces

A **sequence space** is a normed vector space where each element is a sequence of complex numbers. The simplest such space we'll consider is c_0, the set of all **null sequences**, i.e., sequences that converge to 0. Define addition and scalar multiplication on c_0 as follows: for any $x, y \in c_0$, let $(x+y)_n = x_n + y_n$ for each n, and for any $\alpha \in \mathbb{C}$, let $(\alpha x)_n = \alpha x_n$ for each n. The additive identity of c_0 is of course the constant zero sequence. We can define a norm on c_0 by $\|x\| = \max_i |x_i|$. (Any null sequence must have an element of maximal magnitude because otherwise, it would have a subsequence that didn't converge to 0.)

The other sequence spaces of interest are the ℓ_q spaces, where $q \in [1, \infty)$. Each element of ℓ_q is a ***q*-summable sequence**; that is, a sequence x such that the series

$$\sum_{n=1}^{\infty} |x_n|^q$$

converges. Addition and scalar multiplication work as in c_0. The norm of an element is the qth root of the appropriate series, analogous to the Euclidean norm on $\mathbb{R}^n$.

Frequently, we'll examine sequences of points in these spaces; that is, sequences of sequences. This raises notational issues: is "x_2" the second complex number in a vector in c_0 or the second vector of a sequence of vectors in c_0? In general, the names of our vectors won't need subscripts, so we'll use subscripts to index the complex elements of a vector. When we need to talk about sequences of sequences, we'll mention the subscript when we quantify each vector, and we'll use parentheses to access the elements of sequences of sequences, so "$x_2(3)$" will mean the third element of the second vector in the sequence (x_n).

A particularly important property of c_0 and ℓ_q is **completeness**, the fact that every Cauchy sequence (of sequences!) in such a space converges to a point (i.e. sequence) in the same space. (See Carothers (2000) for proofs, which are somewhat lengthy and hence omitted here.) Thus c_0 and ℓ_q are complete normed vector spaces, or **Banach spaces** for short. Henceforth, any unquantified use of "X" should be taken to refer to one of these spaces. In general, we'll consider all these Banach spaces together as much as possible, and we'll be able to obtain the same results for all of them, but we'll need to work harder in the case of ℓ_q than c_0.

The proofs of the following lemmas, which we'll need later, help give a sense of the character of these spaces.

Lemma 2.1. For any $x \in X$ and $\epsilon > 0$, there exists n with $\|x - (x_1, x_2, \ldots, x_n, 0, 0, \ldots)\| < \epsilon$.

Proof. For $X = c_0$, choose n such that $|x_k| < \epsilon$ for all $k > n$. Then

$$\|x - (x_1, x_2, \ldots, x_n, 0, 0, \ldots)\| = \max\{0, |x_{n+1}|, |x_{n+2}|, \ldots\} < \epsilon .$$

For $X = \ell_q$, the series $\sum_{i=1}^{\infty} |x_i|^q$ converges absolutely, so there exists n with $\sum_{i=k}^{\infty} |x_i|^q < \epsilon^q$ for all $k > n$. Thus

$$\|x - (x_1, x_2, \ldots, x_n, 0, 0, \ldots)\| = \sqrt[q]{\sum_{i=n+1}^{\infty} |x_i|^q} < \sqrt[q]{\epsilon^q} = \epsilon . \qquad \square$$

Lemma 2.2 will help us reason about c_0 and ℓ_q simultaneously.

Lemma 2.2. For any $x \in X$, $\|x\| \leq \|x\|_1$, where $\|x\|_1$ is defined to be $\sum_{n=1}^{\infty} |x_n|$, the norm of ℓ_1.

Proof. Notice that $\|x\|_1$ may not converge. For example, $\|x\|_1$ is the harmonic series when $x = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$, which is an element of c_0 and ℓ_2. But divergence is fine, since the sequence of partial sums of $\|x\|_1$ is nondecreasing, and hence $\|x\|_1$ must diverge to $+\infty$ if it fails to converge. So suppose $\|x\|_1$ converges.

For $X = c_0$, choose an $n_0 \in \mathbb{N}$ that maximizes $|x_{n_0}|$. Then

$$\|x\| = |x_{n_0}| \leq |x_1| + |x_2| + \cdots + |x_{n_0}| + \cdots = \|x\|_1 \ .$$

For $X = \ell_q$, we prove inductively that $\sum_{k=1}^{n} |x_k|^q \leq (\sum_{k=1}^{n} |x_k|)^q$ for all n. The case of 1 is immediate. Assuming that the statement holds for 2 as well as for some $n \geq 2$, the case of $n+1$ is also easy:

$$\left(\sum_{k=1}^{n+1} |x_k|\right)^q \geq \left|\sum_{k=1}^{n} |x_k|\right|^q + |x_{n+1}|^q \geq \sum_{k=1}^{n} |x_k|^q + |x_{n+1}|^q = \sum_{k=1}^{n+1} |x_k|^q \ .$$

All that remains is the case of 2. Define $f : [0, \infty) \to \mathbb{R}$ by

$$f(t) = (|x_1| + t)^q - (|x_1|^q + t^q) \ .$$

Then f' is given by

$$f'(t) = q(|x_1| + t)^{q-1} - qt^{q-1} = q[\ (|x_1| + t)^{q-1} - t^{q-1}\ ] \ .$$

Since $q \geq 1$, $q - 1 \geq 0$, so $f'(t) \geq 0$ for all $t \in [0, \infty)$. Thus f is nondecreasing on $[0, \infty)$. Since $f(0) = 0$, f is nonnegative on $[0, \infty)$. This means that $(|x_1| + t)^q \geq |x_1|^q + t^q$ for all $t \geq 0$. In particular, $(|x_1| + |x_2|)^q \geq |x_1|^q + |x_2|^q$.

Hence, for all n, $\sum_{k=1}^{n} |x_k|^q \leq (\sum_{k=1}^{n} |x_k|)^q$ and thus $\sqrt[q]{\sum_{k=1}^{n} |x_k|^q} \leq \sum_{k=1}^{n} |x_k|$. It follows that $\|x\| \leq \|x\|_1$. □

Lemma 2.3. X is **perfect**; that is, X has no singleton open subsets.

Proof. Let $x \in X$ and let U be a neighborhood of x. Choose $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. Define y to be the same as x except set $y_1 = x_1 + \frac{\epsilon}{2}$. Clearly, $y \in X$ but $y \neq x$. Then by

Lemma 2.2,

$$\|x-y\| \leq |x_1-y_1| + \sum_{n=2}^{\infty} |x_n-y_n| = \left|x_1 - x_1 - \tfrac{\epsilon}{2}\right| + 0 < \epsilon\ ,$$

so $y \in U$. □

> **Lemma 2.4.** X is **separable**; that is, X has a countable dense subset.

Proof. Define

$$A = \bigcup_{N\in\mathbb{N}} \{(a_1+b_1 i,\ a_2+b_2 i,\ \ldots) : a_n, b_n \in \mathbb{Q}\ \forall n,\ a_n = b_n = 0\ \forall n > N\}\ .$$

Since A is a countable union of countable sets, it is itself countable. Since every element of A has an infinite tail of 0s, $A \subseteq X$.

Choose $x_0 \in X$. For each n, define the sequence of complex numbers x_n such that for all $k \leq n$, the real and imaginary parts of $x_n(k)$ are rational but $|x_n(k) - x_0(k)| < \frac{1}{n^2}$, and for all other k, $x_n(k) = 0$. Clearly, $x_n \in A$ for all n. So let $\epsilon > 0$.

Choose $N_1 > \frac{2}{\epsilon}$. Then $\frac{1}{N_1} < \frac{\epsilon}{2}$. Use Lemma 2.1 to find N_2 such that $\|x_0 - (x_0(1), x_0(2), \ldots, x_0(N_2), 0, \ldots)\| < \frac{\epsilon}{2}$. Then for all $n > \max\{N_1, N_2\}$, by Lemma 2.2,

$$\begin{aligned}
\|x_0 - x_n\| &\leq \|x_0 - (x_0(1), x_0(2), \ldots, x_0(n), 0, \ldots)\| \\
&\qquad + \|(x_0(1), x_0(2), \ldots, x_0(n), 0, \ldots) - x_n\| \\
&\leq \|x_0 - (x_0(1), x_0(2), \ldots, x_0(N_2), 0, \ldots)\| + \sum_{k=1}^{n} |x_0(k) - x_n(k)| \\
&< \frac{\epsilon}{2} + n\frac{1}{n^2} \\
&< \frac{\epsilon}{2} + \frac{1}{N_1} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon\ ,
\end{aligned}$$

so $x_n \to x_0$. Thus, $X \subseteq \overline{A}$, so X is separable. □

Remark 2.5. For any $x, y \in X$, define xy such that $(xy)_n = x_n y_n$ for all n. That X is closed under this notion of multiplication follows from the inequality $\|xy\| \leq \|x\| \, \|y\|$. When $X = c_0$, this is immediate from the fact that $\max_i |x_i y_i| \leq \max_i |x_i| \max_i |y_i|$. When $X = \ell_q$, observe that

$$\|xy\|^q = \sum |x_i|^q \, |y_i|^q \leq \sum \|x\|^q \, |y_i|^q = \|x\|^q \sum |y_i|^q = \|x\|^q \, \|y\|^q \ .$$

Lemma 2.6. Let $x_0, y_0 \in X$. If there exist $x_n, y_n \in X$ for each n such that $x_n \to x_0$ and $y_n \to y_0$, then $(xy)_n \to x_0 y_0$.

Proof. Let $\epsilon > 0$. Since (x_n) converges, it's bounded, so there exists $M > 0$ with $\|x_n\| < M$ for all n. Pick N_y such that $\|y_n - y_0\| < \frac{\epsilon}{2M}$ for all $n > N_y$. Pick N_x such that $\|x_n - x_0\| < \frac{\epsilon}{2\max\{\|y_0\|, 1\}}$ for all $n > N_x$. Then for all $n > \max\{N_x, N_y\}$,

$$\begin{aligned}
\|(xy)_n - x_0 y_0\| &= \|x_n y_n - x_0 y_0\| \\
&\leq \|x_n y_n - x_n y_0\| + \|x_n y_0 - x_0 y_0\| \\
&= \|x_n (y_n - y_0)\| + \|y_0 (x_n - x_0)\| \\
&\leq \|x_n\| \, \|y_n - y_0\| + \|y_0\| \, \|x_n - x_0\| \\
&< M \frac{\epsilon}{2M} + \|y_0\| \frac{\epsilon}{2\max\{\|y_0\|, 1\}} \\
&\leq \epsilon \ .
\end{aligned}$$

□

Notice that this lemma may be easily extended to any finite number of convergent sequences being multiplied together; e.g., if $x_n \to x_0$, $y_n \to y_0$, and $z_n \to z_0$, then $x_n y_n \to x_0 y_0$, so $x_n y_n z_n = (x_n y_n) z_n \to x_0 y_0 z_0$.

3 Chaos

Very roughly, a function mapping a metric space to itself is chaotic if it eventually (given enough repeated applications of the function) sends points that are close together all over the space. There are many competing definitions of chaos. Here, we'll consider two.

Using Peris's (2003) terminology, a function $f : X \to X$ is **AY-chaotic** (chaotic in the sense of Auslander and Yorke) if it's topologically transitive and sensitive. The function is **topologically transitive** if for any open $U, V \subseteq X$, there exist $x \in U$ and $n \in \mathbb{N}$ with $f^n(x) \in V$. The function is **sensitive** (or, to use the standard terminology, the function **has sensitive dependence on initial conditions**) if there exists $\epsilon > 0$ such that for any $x \in X$ and $\delta > 0$ there exists $y \in B_\delta(x)$ and $n \in \mathbb{N}$ with $\|f^n(x) - f^n(y)\| > \epsilon$.

A point x is a **hypercyclic** point of f if the **orbit** of x [the set $\operatorname{Orb}(x)$ defined by $\{x, fx, f^2x, \ldots\}$] is dense. We also call f itself "hypercyclic" if it has a hypercyclic point. The following theorem, cited but not proved in Peris (2003), connects the existence of hypercyclic points to AY-chaos in X.

Theorem 3.1. For any complete, separable, and perfect metric space M, a function $f : M \to M$ is topologically transitive if and only if it has a hypercyclic point.

For the following lemma about hypercyclic points, note that a topological space is **Hausdorff** if every two points have disjoint neighborhoods. Every metric space is Hausdorff because any two points x and y have neighborhoods $B_{\frac{1}{2}d(x,y)}(x)$ and $B_{\frac{1}{2}d(x,y)}(y)$, which are disjoint because if $z \in B_{\frac{1}{2}d(x,y)}(x) \cap B_{\frac{1}{2}d(x,y)}(y)$, then $d(x,y) \leq d(x,z) + d(y,z) < \frac{1}{2}d(x,y) + \frac{1}{2}d(x,y)$, a contradiction.

Lemma 3.2. If H is a perfect Hausdorff space and $f : M \to M$ has a hypercyclic point x, every element of $\operatorname{Orb}(x)$ is also hypercyclic.

Proof. It suffices to show that fx is hypercyclic; the hypercyclicity of $f^n x$ for all $n > 1$ follows easily by induction. Let $U \subseteq H$ be nonempty and open. Let $A = U \setminus \{x\}$. Since H is Hausdorff, $\{x\}^C$ is open, so given that $A = U \cap \{x\}^C$, A is open. Since H is perfect, U must contain a point that isn't x, so A is nonempty. Thus there exists $n \geq 0$ with $f^n x \in A$. But since $f^n x \neq x$, $n > 0$. So $f^n x = f^{n-1}(fx) \in \text{Orb}(fx)$ and hence $\text{Orb}(fx) \cap U$ is nonempty, meaning that fx is hypercyclic. □

A function is **D-chaotic** (that is, chaotic in the sense of Devaney) if it's AY-chaotic and periodic points of f are dense in X. Or one can define D-chaos using only topological transitivity and density of periodic points; in any infinite metric space, these conditions imply sensitivity (Arfer, 2010).

The functions of interest here are a generalization to sequence spaces of the algebraic notion of polynomials over a field. For any $m \in \mathbb{N}$, a function $Q : X \to X$ is an ***m*-homogeneous polynomial** or a **homogeneous polynomial of degree *m*** if there exists a multilinear continuous function $A : X^m \to X$ such that $A(x, x, \ldots, x) = Q(x)$ for all $x \in X$. Thus, homogeneous polynomials are continuous. A general **polynomial of degree *n*** is a function $P : X \to X$ such that $P = \sum_{m=0}^{n} Q_m$ for some $n \in \mathbb{N}$, some 0-homogeneous polynomial Q_0 (i.e. a constant), some 1-homogeneous polynomial Q_1, and so on up to some n-homogeneous polynomial Q_n. Since a general polynomial is a finite sum of homogeneous polynomials, all polynomials are continuous. To demonstrate that this notion of polynomials is a proper generalization, notice that a polynomial of the usual sort on $\mathbb{C}$, defined by $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, is equal to $z \mapsto A_n(z, z, \ldots, z) + A_{n-1}(z, z, \ldots, z) + \cdots + A_1(z) + A_0()$ where for each m, $A_m : \mathbb{C}^m \to \mathbb{C}$ is defined by $A_m(z_1, z_2, \ldots, z_m) = a_m z_1 z_2 \cdots z_m$.

Peris (2003) notes that while there are many examples of hypercyclic m-homogeneous polynomials on Banach spaces for $m = 1$, no such polynomials exist for greater m. Peris finds that removing the requirement of homogeneity changes the situation drastically. Investigating polynomials that can be expressed as a backward shift composed with an operator that applies a fixed complex-valued polynomial to each sequence element, he shows how the dynamics of the complex-valued polynomial can imply not merely hypercyclicity but D-chaos in the sequence space. Peris's proofs are correct but terse. In the proceeding sections, we'll discuss his results and prove them in much greater detail.

4 Polynomials

Henceforth, define $P : X \to X$ such that $(P(x))_n = p(x_{n+1})$ for each n, where $p : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree at least 2 such that $p(0) = 0$. The theorems in Sections 6 and 7, which constitute our ultimate goal, are statements about P. Our goal for this section is to show that P is a polynomial.

Let d be the degree of p. Let $a_0, a_1, \ldots, a_d$ be the coefficients of p. For each $k = 0, \ldots, d$, define $Q_k : X \to X$ such that $(Q_k(x))_n = a_k x_{n+1}^k$ for all n. Fix $k \in \{1, \ldots, d\}$. We want to show that Q_k is a k-homogeneous polynomial.

Define $A : X^k \to X$ by

$$A(x_1, \ldots, x_k) = (a_k x_1(2) x_2(2) \cdots x_k(2),\ \ a_k x_1(3) x_2(3) \cdots x_k(3),\ \ \ldots)\ .$$

By the closure of X under scalar multiplication and Remark 2.5, $A(X^k) \subseteq X$. To prove multilinearity, choose an argument of A to let vary; we may assume, without loss of generality, that we've chosen the first. Then for all $w, y, x_2, x_3, \ldots x_k \in X$ and $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned}
&\alpha A(w, x_2, \ldots, x_k) + \beta A(y, x_2, \ldots, x_k) \\
&= \alpha(a_k w(2) x_2(2) \cdots x_k(2),\ \ a_k w(3) x_2(3) \cdots x_k(3),\ \ \ldots) \\
&\quad + \beta(a_k y(2) x_2(2) \cdots x_k(2),\ \ a_k y(3) x_2(3) \cdots x_k(3),\ \ \ldots) \\
&= (\alpha a_k w(2) x_2(2) \cdots x_k(2) + \beta a_k y(2) x_2(2) \cdots x_k(2),\ \ \ldots) \\
&= \Big(a_k[\alpha w(2) + \beta y(2)] x_2(2) \cdots x_k(2),\ \ \ldots\Big) \\
&= A(\alpha w + \beta y,\ x_2,\ \ldots,\ x_k)\ ,
\end{aligned}$$

so A is multilinear.

We'll show the continuity of A by splitting A into three simpler functions. Define

$M : X^k \to X$ by $M(x_1, x_2, \ldots, x_k) = x_1 x_2 \cdots x_k$, $J : X \to X$ by $J(x) = a_k x$, and $B : X \to X$ by $B(x) = (x_2, x_3, x_4, \ldots)$.

To show that M is continuous, let (χ_n) be a sequence in X^k that converges to some $\chi_0 \in X^k$. [Watch out for the notation here: (χ_n) is a sequence of elements of X^k, χ_1 is an element of X^k, and $\chi_1(1)$ is an element of X.] A finite product of arbitrary metric spaces is a metric space where any sequence (x_n) converges to a point x if and only if the coordinate sequences of (x_n) converge to the corresponding coordinate of x (Carothers, 2000, p. 48). Thus, $\lim_{n\to\infty} \chi_n(i) = \chi_0(i)$ for each $i = 1, 2, \ldots, k$. By Lemma 2.6, $M(\chi_n) \to M(\chi_0)$, so M is continuous.

The continuity of J is obvious.

To show that B is continuous, let (x_n) be a sequence in X that converges to some $x_0 \in X$. Let $\epsilon > 0$. Choose N such that $\|x_n - x_0\| < \epsilon$ for all $n > N$. Then for all $n > N$,

$$\begin{aligned}
\|B(x_n) - B(x_0)\| &= \|(x_n(2) - x_0(2),\ x_n(3) - x_0(3),\ \ldots)\| \\
&= \|(0,\ x_n(2) - x_0(2),\ x_n(3) - x_0(3),\ \ldots)\| \\
&\leq \|(x_n(1) - x_0(1),\ x_n(2) - x_0(2),\ x_n(3) - x_0(3),\ \ldots)\| \\
&= \|x_n - x_0\| \\
&< \epsilon\ .
\end{aligned}$$

Thus $B(x_n) \to B(x_0)$, so B is continuous.

Since $A = B \circ J \circ M$ and B, J, and M are continuous, A is continuous. Since $Q_k(x) = A(x, x, \ldots, x)$ for all $x \in X$, Q_k is a k-homogeneous polynomial. And since $P = \sum_{k=0}^{d} Q_k$, P is a polynomial.

5 Three lemmas in $\mathbb{C}$

We seek to demonstrate the chaotic behavior of P, a polynomial in a sequence space, by referring to the behavior of p, the underlying complex-valued polynomial. To that end, we'll prove some lemmas about complex dynamics.

The following three facts are standard theorems in complex analysis.

Theorem 5.1 (Saff and Snider 2003, p. 378)**.** If f is analytic at a point $z \in \mathbb{C}$ and $f'(z) \neq 0$, there exists a neighborhood U of z such that $f|_U$ is injective.

Open-Mapping Theorem (Saff and Snider 2003, p. 363)**.** If $U \subseteq \mathbb{C}$ is open and f is a non-constant function analytic on U, then $f(U)$ is open.

Jordan Curve Theorem (Saff and Snider 2003, p. 158)**.** Any simple closed curve separates $\mathbb{C}$ into two disjoint open sets, one (the **inside**) bounded and simply connected and the other (the **outside**) unbounded and not simply connected.

(A connected set $S \subseteq \mathbb{C}$ is **simply connected** if for any simple closed curve in S, the inside of the curve is a subset of S.)

We can now prove Lemma 5.2. We'll use it both immediately, for proving Lemma 5.3, and later on. Applying it requires a repelling fixed point. We call a periodic point z of a function f with period k "**repelling**" if f is analytic at z and $\left|(f^k)'z\right| > 1$.

By the way, so as not to confuse balls in X with balls in $\mathbb{C}$, we'll use the notation "$D_\delta(z)$" to refer to the open δ-ball (i.e. open δ-disk) about a point $z \in \mathbb{C}$.

Lemma 5.2. If $u \in \mathbb{C}$ is a repelling fixed point of f, then there exists an open disk U_0 centered at u with $U_0 \subseteq f(U_0)$.

Proof. Since $f'(u) = \lim_{z \to u} \frac{f(z)-f(u)}{z-u}$ and the modulus function is continuous, $|f'(u)| = \lim_{z \to u} \left| \frac{f(z)-f(u)}{z-u} \right|$. Thus there exists a neighborhood U_1 of u with $\left| \frac{f(z)-f(u)}{z-u} \right| > 1$ and hence (since $f(u) = u$)

$$|f(z) - u| > |z - u| \quad \text{for all} \quad z \in U_1 \setminus \{u\} \,. \tag{1}$$

By Theorem 5.1, there exists a neighborhood U_2 of u with $U_2 \subseteq U_1$ and $f|_{U_2}$ injective. By the open-mapping theorem, $f|_{U_2}$ maps open sets to open sets. Thus the preimage of any open set under $f|_{U_2}^{-1}$ is open. Thus $f|_{U_2}$ has a continuous inverse, so $f|_{U_2}$ is a homeomorphism.

Choose $r \in (0, \infty)$ such that $\overline{D_r(u)} \subseteq U_2$. Let $U_0 = D_r(u)$ and let $C_r = \partial U_0$. Then $f(C_r)$ is a closed curve, since $f(C_r) = (f \circ c)(S^1)$ for the continuous injection $c : S^1 \to \mathbb{C}$ defined by $c(\theta) = u + re^{i\theta}$. Since $f|_{U_2}$ is injective and $C_r \subseteq \overline{D_r(u)} \subseteq U_2$, $f(C_r)$ is simple.

Let $z \in U_0$ and suppose for contradiction that $z \notin f(U_0)$. Since $f|_{U_2}$ is a homeomorphism, $f|_{U_2}^{-1}$ sends cutsets to cutsets, so $z \notin f(C_r)$. Thus, by the Jordan curve theorem, $f(z)$ and u are on opposite sides of $f(C_r)$, so the line segment from u to $f(z)$ intersects $f(C_r)$. Then there exists $f(z_c) \in f(C_r)$ with $|f(z_c) - u| < |f(z) - u|$ and thus, by (1), $|z_c - u| < |z - u| < r$, which contradicts the fact that $|z_c - u| = r$. So in fact, $z \in U_0$, so $U_0 \subseteq f(U_0)$. □

The **Julia set** of a function f, denoted "$\mathcal{J}(f)$", is the closure of $\mathcal{R}(f)$, the set of repelling periodic points of f. The Julia set can also be defined as the complement of the **Fatou set** $\mathcal{F}(f)$, the set of points z such that $\{f^n : n \in \mathbb{N}\}$ is normal on some neighborhood of z. A family of analytic functions A is **normal** on an open set U if it's equicontinuous on any compact subset of U. And any family of functions A is **equicontinuous** on a set S if for every $z_0 \in S$ and $\epsilon > 0$ there exists $\delta > 0$ such that $|f(z_0) - f(z)| < \epsilon$ for all $z \in D_\delta(z_0)$ and $f \in A$. Just as uniform continuity is a strengthened form of continuity that produces δs valid for every point in a set, equicontinuity is a strengthened form of continuity that produces δs valid for every function in a family.

We'll need the following theorem about normal families of functions, which Peris (2003) mentions but does not prove.

Montel's Theorem. If there exist three points in $\mathbb{C} \cup \{\infty\}$ that are all omitted from $f(\mathbb{C} \cup \{\infty\})$ for every $f \in A$, then A is normal.

Lemma 5.3 is as powerful is it is surprising. Given a point x_0 in the Julia set of a function f and any old finite collection of points $z_1, z_2, \ldots, z_n$, the lemma harnesses the explosion of f nearby x_0 (as implied by Lemma 5.2) to find points near x_0 that f eventually sends near the z_is.

Lemma 5.3. Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial with degree at least 2. Let $x_0 \in \mathcal{J}(p)$, $U \subseteq \mathbb{C}$ be a neighborhood of x_0, $\delta > 0$, and $z_1, z_2, \ldots, z_n \in \mathbb{C}$. Then there exist $m \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in U$ with $p^m x_i \in D_\delta(z_i)$ for each i.

This statement and its proof are given as Lemma 3.1 in Peris (2003).

Proof. Since $x_0 \in \mathcal{J}(p)$, either x_0 is a limit of a sequence of repelling periodic points or it is itself a repelling periodic point, so there exists a repelling periodic $u \in U$. Let k be the period of u. Since u is a fixed point of p^k, there exists by Lemma 5.2 a disk $U_0 \subseteq U$ centered at u with $U_0 \subseteq p^k(U_0)$. For each $j \in \mathbb{N}$, define $U_j = p^{jk}(U_0)$. Since $U_0 \subseteq p^k(U_0)$,

$$U_j = p^{jk}(U_0) \subseteq p^{jk}(p^k(U_0)) = p^{(j+1)k}(U_0) = U_{j+1} \quad \text{for each} \quad j\,,$$

so the U_is are increasing. Let $A = \{p^{jk}|_{U_0} : j \in \mathbb{N}\}$.

Suppose for contradiction that A is equicontinuous at u. Then setting ϵ to half the radius of U_0, there exists $\Delta > 0$ with $|p^{jk}(z) - p^{jk}(u)| = |p^{jk}(z) - u| < \epsilon$ for all $z \in D_\Delta(u)$ and $j \in \mathbb{N}$. Without loss of generality, we may force $\Delta < \epsilon$. Pick $z_0 \in D_\Delta(u) \setminus \{u\}$. By similar logic as in the proof of Lemma 5.2, there exists $\alpha \in (1, \infty)$ with $|p^k(z) - u| >$

$\alpha |z - u|$ for all $z \in U_0$. Pick n with $\alpha^n |z_0 - u| > \epsilon$. Then

$$\begin{aligned} \epsilon &< \alpha^n |z_0 - u| \\ &< \alpha^{n-1} \left| p^k(z_0) - u \right| \\ &< \alpha^{n-2} \left| p^{2k}(z_0) - u \right| \\ &< \cdots \\ &< \alpha \left| p^{(n-1)k}(z_0) - u \right| \\ &< \left| p^{nk}(z_0) - u \right| \ , \end{aligned}$$

contradicting the choice of Δ. Hence, A isn't equicontinuous.

By the contrapositive of Montel's theorem, A omits at most two points. So for each $i = 1, \ldots, n$, there exists $y_i \in D_\delta(z_i)$ with y_i not omitted by A. For each y_i, pick $j_i \in \mathbb{N}$ so that $y_i \in U_{j_i}$. Then $\{y_1, \ldots, y_n\} \subseteq U_{\max\{j_1,\ldots,j_n\}}$, since the U_is are increasing, so $\{y_1, \ldots, y_n\} \subseteq p^{\max\{j_1,\ldots,j_n\}}(U)$. □

Actually, the preceding proof would work for any entire function p, so the hypothesis that p is a polynomial of degree at least 2 is unnecessarily strong. Assumedly, this hypothesis reflects the fact that Peris (2003) is only concerned with such a p.

Lemma 5.4. If f is entire, then $\mathcal{J}(f) \subseteq \mathcal{J}(f^n)$ for all $n \in \mathbb{N}$.

Proof. We'll prove that $\mathcal{R}(f) \subseteq \mathcal{R}(f^n)$ for all n by induction. Let $n \in \mathbb{N}$ and suppose $\mathcal{R}(f) \subseteq \mathcal{R}(f^n)$. Let $z \in \mathcal{R}(f)$ with period k. Since

$$(f^{n+1})^k z = (f^k)^{n+1} z = f^k(f^k(\cdots f^k(f^k(z))\cdots)) = z \ ,$$

z is a periodic point of f^{n+1}. Since $z \in \mathcal{R}(f) \subseteq \mathcal{R}(f^n)$, $\left|((f^n)^k)'z\right| > 1$, so by the chain rule,

$$\begin{aligned}\left|((f^{n+1})^k)'z\right| &= \left|(f^k \circ f^{nk})'z\right| \\ &= \left|(f^k)'(f^{nk}z) \cdot (f^{nk})'z\right| \\ &= \left|(f^k)'z\right| \left|(f^{nk})'z\right| \\ &> 1 \cdot 1 \\ &= 1 \; .\end{aligned}$$

Thus, $z \in \mathcal{R}(f^{n+1})$.

Since $A \subseteq \overline{A}$ for any set A and $\mathcal{R}(f) \subseteq \mathcal{R}(f^n)$ for all n, $\mathcal{J}(f) \subseteq \mathcal{J}(f^n)$ for all n. □

6 AY-chaotic polynomials on X

Theorem 6.1. The following conditions are equivalent:

(i) P is AY-chaotic.

(ii) P is hypercyclic.

(iii) P is sensitive.

(iv) $0 \in \mathcal{J}(p)$.

This statement and its proof are given as Theorem 3.2 in Peris (2003).

Proof.

(i) $\implies$ (ii) Any AY-chaotic function is by definition transitive. Since X is complete, separable, and perfect, P is hypercyclic by Theorem 3.1.

(ii) $\implies$ (iii) Any ϵ will work, so let's just use 1. Let $x \in X$ and $\delta > 0$ and use Lemma 2.1 to find $k \in \mathbb{N}$ with $\chi = (x_1, x_2, \ldots, x_k, 0, 0 \ldots)$ such that $\|x - \chi\| < \delta$. Notice that

$$\begin{aligned}
P^k\chi &= P^{k-1}(p(x_2),\ p(x_3),\ \ldots,\ p(x_k),\ p(0),\ p(0),\ \ldots) \\
&= P^{k-1}(p(x_2),\ p(x_3),\ \ldots,\ p(x_k),\ 0,\ 0,\ \ldots) \\
&= P^{k-2}(p^2(x_3),\ p^2(x_4),\ \ldots,\ p^2(x_k),\ p(0),\ p(0),\ \ldots) \\
&= P^{k-2}(p^2(x_3),\ p^2(x_4),\ \ldots,\ p^2(x_k),\ 0,\ 0,\ \ldots) \\
&\ \ \vdots \\
&= \vec{0}\,.
\end{aligned}$$

Let h be a hypercyclic point of P. Then there exists $d \in \operatorname{Orb}(h) \cap B_\delta(x)$. By Lemma 3.2, d and $P^k d$ are also hypercyclic. Then $\operatorname{Orb}\left(P^k d\right)$ is dense in X, so there exists $n \in$

$\mathbb{N}$ with $P^{k+n}d \in B_1(w)$ where $w = (3, 0, 0, 0, \ldots)$. Since $\|w\| = 3$ and $\left\|w - P^{k+n}d\right\| < 1$, $\left\|P^{k+n}d\right\| > 2$. Thus,

$$\begin{aligned} 2 &< \left\|\vec{0} - P^{k+n}d\right\| \\ &= \left\|P^n\vec{0} - P^{k+n}d\right\| \\ &= \left\|P^n\left(P^k\chi\right) - P^{k+n}d\right\| \\ &= \left\|P^{k+n}\chi - P^{k+n}d\right\| , \end{aligned}$$

so by the triangle inequality, $\left\|P^{k+n}x - P^{k+n}\chi\right\| > 1$ or $\left\|P^{k+n}x - P^{k+n}d\right\| > 1$.

(iii) $\implies$ **(iv)** Suppose for contradiction that $0 \notin \mathcal{J}(p)$. Then $0 \in \mathcal{F}(p)$, so (p^n) is normal on $D_\delta(0)$ for some $\delta > 0$. We now use three standard theorems in immediate succession:

- The set of derivatives of a set of normal functions is normal (Conway, 1978, p. 154). Thus, $((p^n)')$ is normal.
- A family of complex-valued functions is normal if and only if it's locally bounded (Conway, 1978, p. 153).* Thus, $((p^n)')$ is bounded on $D_\delta(0)$.
- Any compex-valued function with a bounded derivative on an open disk is Lipschitz on that disk (Saff and Snider, 2003, p. 180). Thus, p^n is Lipschitz on $D_\delta(0)$ for each n.

So there exists $M \in (0, \infty)$ such that for all $n \in \mathbb{N}$,

$$\frac{|p^n z_1 - p^n z_2|}{|z_1 - z_2|} < M \quad \text{for all} \quad z_1, z_2 \in D_\delta(0)$$

* This statement is equivalent to Montel's theorem. In fact, it's a more conventional statement of Montel's theorem than the one I gave in Section 5.

and since 0 is a fixed point of p,

$$\frac{|p^n z|}{|z|} < M \quad \text{for all} \quad z \in D_\delta(0) \; .$$

So $|p^n z| < M\,|z|$ for all $n \in \mathbb{N}$ and $z \in D_\delta(0)$. Thus $\|P^n x\| \le M\,\|x\|$ for all $n \in \mathbb{N}$ and $x \in B_\delta(\vec{0})$, contradicting the hypothesis that P is sensitive.

(iv) $\Longrightarrow$ **(i)** Suppose we knew (ii). Then (iii) would follow, as we just proved, and applying Theorem 3.1 would get us (i). So let's prove (ii). Applying Theorem 3.1 again, we need only show transitivity. Then let $x, y \in X$ and $\epsilon > 0$. By Lemma 2.1, pick m_x, m_y with $\|x - (x_1, x_2, \ldots, x_{m_x}, 0, \ldots)\| < \frac{\epsilon}{2}$ and $\left\|y - (y_1, y_2, \ldots, y_{m_y}, 0, \ldots)\right\| < \frac{\epsilon}{2}$ and set $m = \max\{m_x, m_y\}$.

By Lemma 5.4, $0 \in \mathcal{J}(p^{m+1})$. So apply Lemma 5.3 with $\delta = \frac{\epsilon}{2m}$ to the polynomial p^{m+1}, the neighborhood $D_\delta(0)$ of the point 0, and the points $x_1, \ldots, x_m$ to get $n \in \mathbb{N}$ and $w_1, \ldots, w_m \in D_\delta(0)$ with $\left|p^{n(m+1)} w_i - x_i\right| < \delta$ for each i. Define

$$\psi = \{y_1, \ldots, y_m, 0, \ldots, 0, w_1, \ldots, w_m, 0 \ldots\} \; ,$$

where w_1 is at index $n(m+1)+1$. (The reason we used p^{m+1} instead of p was just to ensure that the w_is begin after the y_is end.) Then by Lemma 2.2,

$$\begin{aligned}
\|y - \psi\| &\le \|y - (y_1, \ldots, y_m, 0, \ldots)\| + \|(y_1, \ldots, y_m, 0, \ldots) - \psi\| \\
&< \frac{\epsilon}{2} + \|(0, \ldots, 0, -w_1, \ldots, -w_m, 0, \ldots)\| \\
&\le \frac{\epsilon}{2} + |w_1| + \cdots + |w_m| \\
&< \frac{\epsilon}{2} + m\frac{\epsilon}{2m} \\
&= \epsilon
\end{aligned}$$

and

$$\begin{aligned}\left\|x - P^{n(m+1)}\psi\right\| &= \left\|(x_1 - p^{n(m+1)}w_1,\ \ldots,\ x_m - p^{n(m+1)}w_m,\ 0,\ \ldots)\right\| \\ &\leq \left|x_1 - p^{n(m+1)}w_1\right| + \cdots + \left|x_m - p^{n(m+1)}w_m\right| \\ &< m\frac{\epsilon}{2m} \\ &< \epsilon\ . \qquad \square\end{aligned}$$

❼ D-chaotic polynomials on X

Theorem 7.1. If $0 \in \mathcal{R}(p)$, then P is D-chaotic.

This statement and its proof are given as Proposition 3.3 in Peris (2003).

Proof. Since $0 \in \mathcal{R}(p)$, $0 \in \mathcal{J}(p)$, so by the previous theorem, P is AY-chaotic, and hence it suffices to demonstrate that periodic points of P are dense in X. Let $x \in X$ and $\epsilon > 0$. Choose a $\lambda \in (1, \ |p'(0)|)$. By Lemma 2.1, there exists m with $\|x - (x_1, x_2, \ldots, x_m, 0, \ldots)\| < \frac{\epsilon}{2}$. Lemma 5.2 applied to the function $\frac{1}{\lambda}p$ implies that there there exists $\delta < \frac{\epsilon}{6m}$ with

$$\lambda D_\Delta(0) \subseteq p(D_\Delta(0)) \quad \text{for all} \quad \Delta \in (0, \delta) \ . \tag{2}$$

By Lemma 5.4, $0 \in \mathcal{J}(p^m)$, so apply Lemma 5.3 with the given δ to the polynomial p^m, the neighborhood $D_\delta(0)$ of the point 0, and the points $x_1, \ldots, x_m$ to get $k \in \mathbb{N}$ and $w_{1,1}, w_{1,2}, \ldots, w_{1,m} \in D_\delta(0)$ with $\left|p^{km} w_{1,i} - x_i\right| < \delta$ for each i. For each $i = 1, \ldots, m$, define $w_{0,i} = p^{km} w_{1,i}$.

Since $\lambda > 1$, $\lim_{j\to\infty} \frac{1}{1-\lambda^{-j}} = 1$, and so we may force k to be large enough that $\frac{1}{1-\lambda^{-km}} < 2$. Then $1 > \frac{1}{1-\lambda^{-km}} - 1 = \sum_{j=0}^{\infty} \lambda^{-jkm} - 1 = \sum_{j=1}^{\infty} \lambda^{-jkm}$.

Now let $j \in \mathbb{N}$ and suppose that for each $i = 1, \ldots, m$ we've chosen $w_{j,i} \in D_{\lambda^{-(j-1)km}\delta}(0)$ with $p^{km} w_{j,i} = w_{j-1,i}$. Then for each i, $w_{j,i} \in \lambda^{km} D_{\lambda^{-jkm}\delta}(0)$ and so by (2) $w_{j,i} \in p^{km}(D_{\lambda^{-jkm}\delta}(0))$. So for each i there exists $w_{j+1,i} \in D_{\lambda^{-jkm}\delta}(0)$ with $p^{km} w_{j+1,i} = w_{j,i}$. By induction, a $w_{j,i}$ of this form exists for every $j \in \mathbb{N}$ and $i = 1, \ldots, m$.

Now define

$$\psi = (w_{0,1}, w_{0,2}, \ldots, w_{0,m}, 0, 0, \ldots, 0, w_{1,1}, w_{1,2}, \ldots, w_{1,m}, 0, 0, \ldots) \ ,$$

where for each $j \geq 0$, $w_{j,1}$ is at index $jkm + 1$ of ψ. By construction, ψ is a periodic point of P; its period is km. By Lemma 2.2,

$$\begin{aligned}
\|x - \psi\| &\leq \|x - (x_1, \ldots, x_m, 0, \ldots)\| + \|(x_1, \ldots, x_m, 0, \ldots) - \psi\| \\
&< \frac{\epsilon}{2} + \sum_{i=1}^{m} |x_i - w_{0,i}| + \sum_{j=1}^{\infty} \sum_{i=1}^{m} |w_{j,i}| \\
&< \frac{\epsilon}{2} + m\delta + \sum_{j=1}^{\infty} m\lambda^{-(j-1)km}\delta \\
&= \frac{\epsilon}{2} + m\delta + m\lambda^{0}\delta + \sum_{j=1}^{\infty} m\lambda^{-jkm}\delta \\
&< \frac{\epsilon}{2} + m\frac{\epsilon}{6m} + m\frac{\epsilon}{6m} + m\frac{\epsilon}{6m}\sum_{j=1}^{\infty} \lambda^{-jkm} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6}\sum_{j=1}^{\infty} \lambda^{-jkm} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} \\
&= \epsilon\ .
\end{aligned}$$

Thus, periodic points of P are dense in X. □

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